

# Dynamic Olley-Pakes Productivity Decomposition with Entry and Exit\*

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## Abstract

In this paper, we propose an extension of the productivity decomposition method developed by Olley & Pakes (1996). This extension provides an accounting for the contributions of both firm entry and exit to aggregate productivity changes. It breaks down the contribution of surviving firms into a component accounting for changes in the firm-level distribution of productivity and another accounting for market share reallocations among those firms – following the same methodology as the one proposed by Olley & Pakes (1996). We argue that the other decompositions that break-down aggregate productivity changes into these same four components introduce some biases in the measurement of the contributions of entry and exit.

We apply our proposed decomposition to the large measured increases of productivity in Slovenian manufacturing during the 1995-2000 period – and contrast our results with those of other decompositions. We find that, over a 5 year period, the measurement bias associated with entry and exit is substantial, accounting for up to 10 percentage points of aggregate productivity growth. We also find that market share reallocations among surviving firms played a much more important role in driving aggregate productivity changes.

*Keywords:* decomposition, aggregate productivity, distributional moments

JEL Classification Numbers: C10, O47

## 1 Introduction

Aggregate productivity is a weighted average of productivity at the producer level (firm or plants). Empirically, these producers have vastly different productivity levels, even when the aggregation occurs over narrowly defined sectors. Aggregate productivity changes over time then need not only reflect shifts in the distribution of producer-level productivity. Holding this distribution fixed, aggregate productivity can also change due to composition changes between firms: Due to changes

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in market shares among surviving firms, but also due to the entry of new producers and the exit of old ones.<sup>1</sup> Empirical studies spanning many different countries, industries, and time horizons have consistently shown that those composition changes are an important driver of aggregate productivity changes (See Foster, Haltiwanger and Krizan, 2001; and Bartelsman, Haltiwanger and Scarpetta, 2009). This finding has spurred the development of productivity decomposition methods that can break down aggregate productivity changes into those 4 different components (productivity distribution shifts among survivors, market share reallocations among survivors, entry, and exit). Several of these decomposition methods are based on following individual producers from one period to the next, tracking changes in both their market shares and their productivity (with the exception of entrants and exiters whose productivity in one of the periods cannot be observed). One notable exception is a decomposition based on moments of the joint distribution of market shares and productivity developed by Olley and Pakes (1996) – hereafter OP. That method does not accommodate entry and exit, in the sense that it does not decompose aggregate productivity changes into components that are driven by entry and exit. For a given set of firms, the weighted average of productivity is decomposed into a moment of the firm productivity distribution (the unweighted mean), and a moment of the joint distribution with market shares (the covariance between productivity and market shares).

In this paper, we extend the OP decomposition to also measure the contributions of entry and exit. We argue that this extension eliminates biases in the measurement of those entry/exit contributions that are a feature of the other decomposition methods that follow individual producers over time. Empirically, we show that these biases are substantial for the case of Slovenia’s transition period from 1995-2000, especially when considering longer time spans. Our empirical results also show that the contribution of market share reallocations among surviving firms to productivity growth is much more substantial, once the biases regarding entry and exit measurement are eliminated.

Setting aside the measurement of entry and exit, the OP decomposition has an attractive feature relative to the decompositions that track individual firms over time: Because it is based on moments of the distributions of productivity and market shares, it can be more directly connected to theoretical models with firm productivity heterogeneity that have been developed to analyze the

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<sup>1</sup>Entry and exit are very similar to market share changes involving a change from or to a zero market share; but there is one substantial difference. One can observe the productivity of a surviving firm as its market share changes; the productivity of a producer, prior to entry or subsequent to exit, cannot be observed.

pattern of market share reallocations across firms and its consequences for aggregate productivity.<sup>2</sup> Given a distribution of firm productivity, these models feature a market mechanism that determines an allocation of market shares to those firms based on their productivity and other firm and market characteristics. This implies a given covariance between those market shares and firm productivity – which is one of the key moments tracked by the OP decomposition. The other moment, the unweighted productivity mean, tracks shifts in the distribution of productivity. (Of course, higher moments of both the productivity distribution and its joint distribution with market shares may also be important for the theoretical models, but nailing down the first moments is a necessary first step.) Historically, the analysis of aggregate productivity changes across countries and over time has focused on the distribution of firm productivity (centered at its unweighted mean). However, much of the recent literature has shown that differences in the market-share covariance account for substantial portions of those aggregate productivity changes (both over time and across countries).

Since our goal is to highlight differences in decomposition methods, we have picked an empirical case of an economy in transition that exhibits very large productivity changes: depending on the measure of productivity used (labor or TFP), aggregate manufacturing productivity in Slovenia increased between 40% and 50% from 1995 to 2000 (See Polanec, 2004). There is thus a substantial productivity change to be decomposed into the 4 components we have discussed (productivity distribution shifts among survivors, market share reallocations among survivors, entry, and exit). We contrast the results from our decomposition method with the two main methods that are currently used to break down productivity changes into those same 4 components: One by Griliches & Regev (1995) – hereafter GR – and the other by Foster *et al.* (2001) – hereafter FHK. Both methods are refinements of a decomposition developed by Baily, Hulten and Campbell (1992) – hereafter BHC, who were the first to report a break down of aggregate productivity change into those 4 components for U.S. manufacturing. These decomposition methods all produce different measures for the same four components of aggregate productivity. This has induced some discussion detailing the sources of those differences. In this paper, we argue that all those decompositions suffer from some biases that stem from their construction method. In general, the theoretical direction of those biases (as well as their magnitudes) is ambiguous. For the case of a fast growing economy (such as Slovenia from 1995-2000), we show theoretically that this bias involves an over-measurement of the contribution of entry. Our empirical decompositions show that the magnitude

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<sup>2</sup>See, for example, Bartelsman *et al.* (2009), Hsieh and Klenow (2009), and Collard-Wexler, Asker and De Loecker (2011).

of this bias is substantial and is then also reflected in a substantial under-measurement of the contribution of surviving firms to aggregate productivity growth. In particular, the component reflecting market share reallocations among surviving firms is most severely under-measured: we find that its contribution to productivity growth is 2-3 times larger than in the GR and FHK decompositions.

The remainder of the paper is organized as follows: In the next section, we review the existing decompositions for aggregate productivity. In the following section, we define our new decomposition and discuss the key measurement differences with the other decompositions. In the fourth section, we empirically decompose Slovenia’s substantial productivity growth, contrasting the results obtained from our decomposition with the ones from the GR and FHK decompositions. The fifth and last section concludes.

## 2 Review of Existing Decompositions

All methods start with a definition of aggregate productivity at time  $t$  as a share-weighted average of firm productivity  $\varphi_{it}$ :

$$\Phi_t = \sum_i s_{it} \varphi_{it}, \tag{1}$$

where the shares  $s_{it} \geq 0$  sum to 1. The key variable of interest is the change in aggregate productivity over time (from  $t = 1$  to 2)  $\Delta\Phi = \Phi_2 - \Phi_1$ . There are many potential choices for the data counterparts representing the share weight  $s_{it}$  and productivity measure  $\varphi_{it}$ . We review those choices later with our empirical application. For now, we note that our decomposition does not depend on any particular choice of weight or productivity measure. Our starting point is that those choices are such that  $\Delta\Phi$  captures a meaningful dimension of aggregate productivity change (which we seek to decompose). Since this productivity change is measured in differences, we assume that the underlying productivity measure  $\varphi_{it}$  is in logs – so  $\Delta\Phi$  represents a percentage change. In the appendix, we develop an alternate version of our decomposition that applies to productivity measures in levels.<sup>3</sup>

In their seminal contribution, Baily *et al.* (1992) follow surviving firms over time, tracking their changes to both shares  $s_{it}$  and productivity  $\varphi_{it}$ . The change in weighted productivity for

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<sup>3</sup>The existing literature on productivity decompositions typically uses a productivity measure in logs. The advantage of the firm-level measure in levels is that the aggregate productivity measure can have a direct data counterpart. For example, if productivity is measured as valued-added per worker and the weights are employment shares, then  $\Phi_t$  measures aggregate value added per worker.

the surviving firms can then be decomposed into a sum of the productivity changes holding the firms' shares constant (within-firm component) and a sum of the share changes holding the firms' productivity constant (between-firm component). Two additional terms capture the contribution of the set of entering firms and the set of exiting firms. The resulting decomposition of the aggregate productivity change is:

$$\begin{aligned}\Delta\Phi &= \sum_{i \in S} s_{i1}(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})\varphi_{i2} + \sum_{i \in E} s_{i2}\varphi_{i2} - \sum_{i \in X} s_{i1}\varphi_{i1} \\ &= \sum_{i \in S} s_{i1}(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})\varphi_{i2} + s_{E2}\Phi_{E2} - s_{X1}\Phi_{X1},\end{aligned}\tag{2}$$

where  $S$ ,  $E$  and  $X$  denote the sets of surviving, entering and exiting firms. The within-firm component is the first term on the right hand side of (2). It aims to capture the contribution of productivity improvements by surviving firms – that is shifts in the distribution of productivity among surviving firms. The second term is the between-firm component, which seeks to capture the contribution of market share reallocations among surviving firms. The last two components capture the contributions of entry and exit. In the second line, we have re-written those components in terms of the aggregate share of entrants (in period 2)  $s_{E2} = \sum_{i \in E} s_{i2}$  and their aggregate productivity  $\Phi_{E2} = \sum_{i \in E} (s_{i2}/s_{E2})\varphi_{i2}$ ; and the aggregate share of exiters (in period 1)  $s_{X1} = \sum_{i \in X} s_{i1}$  and their aggregate productivity  $\Phi_{X1} = \sum_{i \in X} (s_{i1}/s_{X1})\varphi_{i1}$ . Here, we have used the same definition of aggregate productivity (1), but restricted to a subset of firms.

The GR and FHK decompositions use the same approach as the BHC decomposition: they follow surviving firms over time, tracking both share and productivity changes. The main difference with BHC is that both methods introduce a reference average productivity level. This reference productivity level is used as a benchmark to evaluate the contributions of entrants and exiters relative to surviving firms.

Griliches and Regev (1995) use the average aggregate productivity level between the two periods,  $\bar{\Phi} = (\Phi_1 + \Phi_2)/2$ , as the reference productivity level. Their decomposition is then given by:

$$\begin{aligned}\Delta\Phi &= \sum_{i \in S} \bar{s}_i(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})(\bar{\varphi}_i - \bar{\Phi}) + \sum_{i \in E} s_{i2}(\varphi_{i2} - \bar{\Phi}) - \sum_{i \in X} s_{i1}(\varphi_{i1} - \bar{\Phi}) \\ &= \sum_{i \in S} \bar{s}_i(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})(\bar{\varphi}_i - \bar{\Phi}) + s_{E2}(\Phi_{E2} - \bar{\Phi}) - s_{X1}(\Phi_{X1} - \bar{\Phi}).\end{aligned}\tag{3}$$

The average firm share and productivity,  $\bar{s}_i = (s_{i1} + s_{i2})/2$  and  $\bar{\varphi}_i = (\varphi_{i1} + \varphi_{i2})/2$ , are constructed in the same way as the average aggregate productivity  $\bar{\Phi}$ . This decomposition contains the same four components as (2). The second line re-writes the contributions of entry and exit in terms of the aggregate shares and aggregate productivity levels. This clearly shows how the introduction of the reference productivity level  $\bar{\Phi}$  directly impacts those measured contributions. In the BHC decomposition, the contribution of entry is always positive – regardless of the aggregate productivity of entrants – and the contribution of exit is always negative – again regardless of the aggregate productivity of exiters. The GR decomposition more accurately reflects the fact that the contribution of entrants and exiters to aggregate productivity changes can be positive or negative, depending on whether the aggregate productivity of that subset of firms is above or below the overall aggregate productivity level. In the next section, we will argue that the GR decomposition still introduces some bias into the measurement of the contributions of entry and exit (and hence also to the contribution of surviving firms) to aggregate productivity changes.

Foster *et al.* (2001) use the aggregate productivity level in period 1 instead of the time average  $\bar{\Phi}$  as a reference productivity level. Their decomposition is then given by:

$$\begin{aligned}
\Delta\Phi &= \sum_{i \in S} s_{i1}(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})(\varphi_{i1} - \Phi_1) + \sum_{i \in S} (s_{i2} - s_{i1})(\varphi_{i2} - \varphi_{i1}) \\
&\quad + \sum_{i \in E} s_{i2}(\varphi_{i2} - \Phi_1) - \sum_{i \in X} s_{i1}(\varphi_{i1} - \Phi_1) \quad (4) \\
&= \sum_{i \in S} s_{i1}(\varphi_{i2} - \varphi_{i1}) + \sum_{i \in S} (s_{i2} - s_{i1})(\varphi_{i1} - \Phi_1) + \sum_{i \in S} (s_{i2} - s_{i1})(\varphi_{i2} - \varphi_{i1}) \\
&\quad + s_{E2}(\Phi_{E2} - \Phi_1) - s_{X1}(\Phi_{X1} - \Phi_1).
\end{aligned}$$

The first line of the FHK decomposition captures the contribution of surviving firms to productivity changes. In addition to the within and between firm components (first two terms) there is an additional third component (labeled as a “cross” firm component) that captures the covariance between changes in market share and changes in productivity. The second line captures the contributions of entry and exit, and can be re-written in terms of the aggregate shares and productivity levels as  $s_{E2}(\Phi_{E2} - \Phi_1) - s_{X1}(\Phi_{X1} - \Phi_1)$ , as shown on the fourth line. As with the GR decomposition, those contributions can be positive or negative, although their sign now depends on whether the aggregate productivity of entrants and exiters is above or below the aggregate productivity in period 1. This also attenuates the bias previously mentioned with respect to the BHC decomposition,

but we again will argue that some bias still remains.

The other commonly used decomposition proposed by Olley & Pakes (1996) eschews following firms over time and instead is based on a decomposition of the aggregate productivity level  $\Phi_t$  in each period. This decomposition is:

$$\begin{aligned}\Phi_t &= \bar{\varphi}_t + \sum_i (s_{it} - \bar{s}_t)(\varphi_{it} - \bar{\varphi}_t) \\ &= \bar{\varphi}_t + \text{cov}(s_{it}, \varphi_{it}),\end{aligned}\tag{5}$$

where  $\bar{\varphi}_t = \frac{1}{n_t} \sum_{i=1}^{n_t} \varphi_{it}$  is the unweighted firm productivity mean and  $\bar{s}_t = 1/n_t$  is the mean market share. We have introduced a slight abuse of notation with the definition of the cov operator, which would typically be multiplied by  $1/n_t$ . However, since  $s_{it}$ s are market shares, they essentially already incorporate the division by the number of firms  $n_t$ . Changes in productivity over time  $\Delta\Phi$  are then simply given by the change in the unweighted mean  $\Delta\bar{\varphi}$  and the change in covariance  $\Delta\text{cov}$ . This provides a natural way of decomposing productivity changes into a component capturing shifts in the productivity distribution (via the change in the first moment  $\Delta\bar{\varphi}$ ) and another component capturing market share reallocations via the change in covariance. This term is different than the cross term from the FHK decomposition, which captures the covariance of market share and productivity *changes for an individual firm*. On the other hand, the OP covariance is measured purely in the joint cross-sectional distribution of market shares and productivity: it increases with the correlation between market shares and productivity. In the next section, we extend this decomposition to accommodate entry and exit, and argue that it more accurately captures the separate contributions to aggregate productivity changes of entrants, exiters, and surviving firms. Within these 3 groups of firms, our decomposition preserves the attractive features of the original OP decomposition in providing a natural additional decomposition between shifts in the distribution of productivity and market share reallocations.

### 3 Dynamic Olley-Pakes Decomposition with Entry and Exit

We start by writing aggregate productivity in each period as a function of the aggregate share and aggregate productivity of the three groups of firms (survivors, entrants, and exiters):

$$\begin{aligned}\Phi_1 &= s_{S1}\bar{\Phi}_{S1} + s_{X1}\bar{\Phi}_{X1} = \bar{\Phi}_{S1} + s_{X1}(\bar{\Phi}_{X1} - \bar{\Phi}_{S1}), \\ \Phi_2 &= s_{S2}\bar{\Phi}_{S2} + s_{E2}\bar{\Phi}_{E2} = \bar{\Phi}_{S2} + s_{E2}(\bar{\Phi}_{E2} - \bar{\Phi}_{S2}).\end{aligned}$$

We have used the same definition of aggregate productivity (1) restricted to the set of surviving firms to write their aggregate productivity in each period:  $\Phi_{St} = \sum_{i \in S} (s_{it}/s_{St}) \varphi_{it}$ , and  $s_{St} = \sum_{i \in S} s_{it}$ . We can then write the productivity change  $\Delta\Phi$  in terms of these components and then separately apply the OP decomposition (5)

$$\begin{aligned}\Delta\Phi &= (\bar{\Phi}_{S2} - \bar{\Phi}_{S1}) + s_{E2}(\bar{\Phi}_{E2} - \bar{\Phi}_{S2}) + s_{X1}(\bar{\Phi}_{S1} - \bar{\Phi}_{X1}) \\ &= \Delta\bar{\varphi}_S + \Delta\text{cov}_S + s_{E2}(\bar{\Phi}_{E2} - \bar{\Phi}_{S2}) + s_{X1}(\bar{\Phi}_{S1} - \bar{\Phi}_{X1}).\end{aligned}\tag{6}$$

The first line decomposes the aggregate productivity change into components for the three groups of firms: survivors, entrants, and exiters. In the second line, we apply the OP decomposition (5) to the contribution of surviving firms – further separating that component into one induced by a shift in the distribution of firm productivity (the unweighted mean change in the productivity of surviving firms  $\Delta\bar{\varphi}_S$ ), and another one induced by market share reallocations (the covariance change between market share and productivity for surviving firms  $\Delta\text{cov}_S$ ).<sup>4</sup> We can also further decompose the contributions of entrants and exiters in a similar way.<sup>5</sup>

As our decomposition leverages the cross-sectional OP decomposition, the first step of separating out the contributions by survivors, entrants, and exiters need not use the same reference productivity level for all three groups. This is a necessary feature of the other decompositions that track individual firms over time. Table 1 highlights those differences between our decomposition and the GR and FHK decompositions. It contrasts the contributions of surviving, entering, and

<sup>4</sup>The relevant market share for this covariance term is the firm’s market share *within* the subset of surviving firms (so those market share sum to 1 within this subgroup).

<sup>5</sup>The contribution of entry can be decomposed into one component reflecting differences in the productivity distribution between entrants and surviving firms,  $s_{E2}(\bar{\varphi}_{E2} - \bar{\varphi}_{S2})$ , and another component reflecting differences in the covariance between market shares and productivity for the two groups,  $s_{E2}(\text{cov}_{E2} - \text{cov}_{S2})$ . The decomposition for exit is very similar. We do not show those further decompositions above in order to maintain the parallel with the 4 components of the GR and FHK decompositions.



exiting firms across the three decompositions. The first two columns clearly show how the same reference productivity level must be used to evaluate the contributions of the three groups of firms for the GR and FHK decompositions. In our proposed decomposition (third column), we can use different reference productivity levels for all three groups.

**Table 1: Productivity Contributions of Surviving, Entering, and Exiting Firms**

Group	GR	FHK	Dynamic OP
Surviving Firms	$s_{S2}(\Phi_{S2} - \bar{\Phi}) - s_{S1}(\Phi_{S1} - \bar{\Phi})$	$s_{S2}(\Phi_{S2} - \Phi_1) - s_{S1}(\Phi_{S1} - \Phi_1)$	$\Phi_{S2} - \Phi_{S1}$
Entering Firms	$s_{E2}(\Phi_{E2} - \bar{\Phi})$	$s_{E2}(\Phi_{E2} - \Phi_1)$	$s_{E2}(\Phi_{E2} - \Phi_{S2})$
Exiting Firms	$s_{X1}(\bar{\Phi} - \Phi_{X1})$	$s_{X1}(\Phi_1 - \Phi_{X1})$	$s_{X1}(\Phi_{S1} - \Phi_{X1})$

All three decompositions feature a contribution of entry that increases with the aggregate productivity of entrants  $\Phi_{E2}$ , a contribution of exit that increases with lower aggregate productivity of exiters  $\Phi_{X1}$ , and a contribution of surviving firms that increases with the aggregate productivity difference  $\Phi_{S2} - \Phi_{S1}$ . (Needless to say, they also all three add up to the same aggregate productivity change  $\Delta\Phi$ .) However, we argue that our decomposition more accurately reflects the contributions of those three groups in the sense that we can relate each group contribution to a specific counterfactual scenario as follows: The contribution of surviving firms is simply the aggregate productivity that would have been observed absent entry and exit. Since neither the productivity of entrants in period 1, nor the productivity of exiters in period 2, are observed, we cannot use an identical counterfactual for those two groups of firms. Instead, we can use the set of surviving firms as a benchmark and ask how adding the group of entrants (or exiters) affects the aggregate productivity change. Thus, our contribution of entry,  $s_{E2}(\Phi_{E2} - \Phi_{S2})$ , is the change in aggregate productivity  $\Delta\Phi$  generated by adding/removing the group of entrants. Similarly, our contribution of exit,  $s_{X1}(\Phi_{S1} - \Phi_{X1})$ , is the change in aggregate productivity  $\Delta\Phi$  generated by adding/removing the group of exiting firms.

We note that using a different reference productivity level for entrants and exiters is critical in order to apply this ‘counterfactual’ definition. Entrants generate positive productivity growth if (and only if) they have higher productivity  $\Phi_{E2}$  than the remaining (surviving) firms  $\Phi_{S2}$  in the same time period when entry occurs ( $t = 2$ ); Exiters generate positive productivity growth if (and only if) they have lower productivity  $\Phi_{X1}$  than the remaining (surviving) firms  $\Phi_{S1}$  in the same time

period when exit occurs ( $t = 1$ ). Because the GR and FHK decompositions follow surviving firms over time, they need to use the same reference productivity levels for entrants and exiters. Any choice of reference productivity level will necessarily lead to a bias in measuring the contribution of one group or the other (and potentially to a bias for both groups). When there is productivity growth that is driven by productivity improvements by surviving firms, then the productivity of surviving firms  $\Phi_{S2}$  is higher than both  $\Phi_1$  (the reference productivity level for FHK) and  $\bar{\Phi}$  (the reference productivity level for GR). This implies an over-measurement of the contribution of entry for both decompositions, and hence an under-measurement of the contribution for the remaining two groups of firms. In the following section, we apply all three decompositions to the fast growing Slovenian economy, and show that this over-measurement is quite substantial.

#### 4 Empirical Application: Slovenian Manufacturing 1995-2000

We use accounting firm-level panel data covering the entire Slovenian manufacturing sector (NACE 2-digit industries 15-37) for the 1995-2000 period.<sup>6</sup> During this time, the manufacturing sector (and the rest of the economy) went through significant structural changes that were triggered by economic reforms adopted in the late 1980s and early 1990s (e.g. liberalization of prices and wages, deregulation of firm entry, and privatization of state-owned firms). The shock caused by economic reforms initially led to large declines in both aggregate output and labor productivity, followed by a fast reversal to high growth, which was sustained through our entire sample period. During that time, value added per worker increased nearly 50%. Previous empirical research using Slovenian manufacturing data (Polanec, 2004; Bartelsman, Haltiwanger and Scarpetta 2007) has established that an important part of this aggregate productivity growth was driven by market share reallocation between surviving, entering and exiting firms. This evidence that all four channels contributed to substantial aggregate productivity gains makes our data set ideal for illustrating our new decomposition, and contrasting its predictions with the GR and FHK decompositions.

##### Description of Data

The data set contains information on firm identity, year of reporting, annual sales, costs of material and services, nominal physical capital and employment. From these we can calculate (or estimate) all the standard measures for labor and total factor productivity at the firm level. We deflate firm

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<sup>6</sup>We are grateful to the Slovenian Agency for Public Legal Records and Related Services (AJ PES) for providing the data.

revenue and material costs by the firm's NACE 2-digit producer price index, and physical capital by the price index for investment goods. The reported number of employees is calculated from the annual number of hours worked by all workers.

Table 2 reports summary statistics for the first and last year of our sample (1995 and 2000). In order to include the same set of firms in all decompositions (regardless of the productivity and share measure used), we only keep firms that employ at least one worker, have positive physical capital, and generate positive value added. The remaining number of firms in our sample increases by 18.4% between 1995 and 2000, from 3,867 to 4,580. Among these firms were 2,677 survivors, 1,903 entrants and 1,191 exitors (based on transitions from 1995 to 2000; we break down transitions at intermediate time intervals later on). Over this time, the average size of active firms, measured by employment, decreased from 60.1 to 45.2 employees, mainly driven by the entry of smaller new firms (size reductions by surviving firms played a more minor role).<sup>7</sup> The downsizing of surviving firms and the exit of firms also contributed to a decline in aggregate employment by 11.2%, from 233 to 207 thousand workers. Nevertheless, real aggregate sales, real aggregate value added and real aggregate physical capital all substantially increased over that time span: by 46.1%, 45.8% and 25.3%, respectively.

**Table 2: Descriptive Statistics for Slovenian Manufacturing Firms in 1995 and 2000**

	Year	
	1995	2000
Number of		
All firms	3867	4580
Surviving firms	2677	2677
Entering firms	-	1903
Exiting firms	1191	-

  

Variable	Year	
	1995	2000
Average employment	60.1	45.2
Aggregate employment [in thousand]	232	206
Real aggregate value added [in bln. SIT]	425	620
Real aggregate output [in bln. SIT]	1520	2220
Real aggregate physical capital [in bln. SIT]	862	1080

Notes: The nominal value added and output are deflated by 2-digit NACE industry-level producer price indices. The nominal physical capital is deflated by investment goods price index.

Source: AJPES and own calculations.

<sup>7</sup>The average employment of surviving firms between 1995 and 2000 declined from 67.4 to 59.4 employees. In 2000, the average employment of entrants was 25.1 employees.

## Choice of Productivity Measure and Weights

There are numerous possibilities for the choice of a productivity measure and associated market share weight. Foster *et al.* (2001) discuss how these choices can affect the decomposition of aggregate productivity for a given decomposition method. Our goal is to contrast the different decomposition methods for a given choice of productivity measure and weights. We restrict our analysis to two main productivity measures and associated weights: one measure of labor productivity with employment shares as weights, and one measure of total factor productivity (TFP) with value-added shares as weights. We directly compute labor productivity as the log of value-added per worker. We estimate TFP for our second measure as the residual of the firm-level production function regression:

$$\ln TFP_{it} = \ln Y_{it} - \hat{\alpha} \ln K_{it} - \hat{\beta} \ln L_{it}, \quad (7)$$

where  $Y_{it}$ ,  $K_{it}$  and  $L_{it}$  denote the real value added, real capital and employment of firm  $i$  in period  $t$ , and  $\hat{\alpha}$  and  $\hat{\beta}$  denote the regression coefficients for capital and labor. Our production function regressions also include both 2-digit NACE industry and annual time dummies. We use a value-added production function (and value-added weights) instead of one based on gross output (with intermediate inputs in the production function), as Petrin and Levinsohn (2005) argue that the former yields productivity estimates that have a much more direct welfare interpretation. However, we have also experimented with TFP based on gross output production function regressions and corresponding weights equal to gross output shares, and the qualitative features that we emphasize are robust to this alternative productivity measure.

There is a vast (and growing) literature on productivity estimation that explores many associated measurement and estimation issues. One of those issues is the simultaneity bias between shocks to productivity and changes in variable inputs (typically labor in a value-added production function framework). For our sample, correcting for this bias does not substantially change the labor and capital elasticities, and hence has negligible effects on the productivity rankings and year to year changes that are the key ingredients for our productivity decompositions (along with the shares, which are directly measured). Another important issue is the unobserved firm price, which affects the measures of output (gross or value-added). Foster *et al.* (2008) find that entering firms charge lower prices relative to incumbent firms. This depresses the measurement of the physical output of entrants relative to incumbents. Thus, we should be clear that our productivity measure relates to revenues and not physical output.

More generally, we emphasize again that our focus is on the contrast between decomposition methods for a given set of productivity measures and weights. Addressing the numerous measurement issues for firm productivity will lead to a different starting point for the decompositions; but the contrast between the decompositions that we highlight will remain. Those differences are based on the use of a sample featuring strong productivity growth among incumbent firms. In addition, one of our productivity measures, value added per worker, is directly computed, and thus of direct interest. In the appendix, where we adapt our decomposition to productivity changes in levels (not in logs), the aggregate productivity measure is simply aggregate manufacturing value added per worker. This variable is clearly relevant and important, regardless of the productivity measurement issues discussed above.

## Results

We report the results from the GR and FHK decompositions, and our proposed dynamic OP decomposition with entry and exit in tables 3, 4, and 5. The top panel is based on the labor productivity measure with employment shares, and the bottom panel is based on the TFP measure with value added weights. We report decompositions between 1995 and all subsequent years until 2000 in order to illustrate how the measurement biases are affected by the length of the time span. The right-hand column lists the same aggregate productivity change across all 3 tables; since productivity is measured in logs, those numbers represent percentage changes for the given time span. The productivity contributions in the other columns (also in percentage changes) sum to this right-hand column.

As we compare the contributions of entry across the three tables, we can easily verify the direction of the over-measurement that we previously motivated for the GR and FHK decompositions (relative to our decomposition). All three decompositions roughly measure the same contribution for entry over the 1 year interval from 1995 to 1996: negative 2-3 percentage points for labor productivity and 0 for TFP. However, as the time span increases, we see that the contributions of entry for the GR and FHK decompositions steadily increase, up to 9-10 percentage points for FHK and up to 4-5 percentage points for GR. On the other hand, our decomposition reports that this contribution of entry remains steady at all the different time spans: slightly negative for labor productivity and zero for TFP.

Consider first the case of labor productivity. Table 6 reports the underlying aggregate labor productivity and share data for all three groups of firms (entrants, exiters, and survivors) across

**Table 3: Griliches and Regev Decomposition**

Aggregate Productivity Change Relative to 1995					
Year	Surviving Firms		Entering Firms	Exiting Firms	All Firms
	Within	Between			
<i>Labor Productivity (in logs) – Emp. Share Weights</i>					
1996	0.1050	0.0164	-0.0249	0.0355	0.1320
1997	0.2266	0.0223	-0.0044	0.0663	0.3108
1998	0.2260	0.0252	0.0070	0.0906	0.3488
1999	0.2807	0.0341	0.0173	0.0983	0.4305
2000	0.3095	0.0370	0.0408	0.1160	0.5033
<i>TFP (in logs) – Value-Added Share Weights</i>					
1996	0.0959	0.0178	0.0039	0.0103	0.1279
1997	0.1925	0.0213	0.0159	0.0321	0.2618
1998	0.1812	0.0261	0.0094	0.0426	0.2593
1999	0.2276	0.0475	0.0338	0.0475	0.3565
2000	0.2512	0.0457	0.0456	0.0588	0.4013

Source: AJPEs and own calculations.

all time spans. Our decomposition reports a negative contribution of entry to aggregate labor productivity change because, in any given year, entrants have an aggregate productivity  $\Phi_{E2}$  that is below the aggregate productivity of surviving firms  $\Phi_{S2}$  for that time period (see bottom panel of Table 6). Thus, in all sample years, the entrants's productivity is below the overall aggregate productivity level  $\Phi_2$ : their presence pulls the aggregate productivity level downward. Concurrently, the aggregate productivity of both surviving firms and entrants is steadily growing, year over year. Part of this growth is attributed to the contribution of entry to aggregate productivity growth in the GR and FHK decompositions. (This is why those decompositions report a contribution of entry that is growing with the time-span.) We describe this finding of a large positive contribution of entry for those decompositions as a bias, because it does not reflect the fact that entrants are pushing the aggregate productivity level downward (in any given year). Tables 3-5 show that this bias can be empirically substantial, accounting for 6-12 percentage points of aggregate productivity growth over 5 years.

Consider next the case of TFP. The same reasoning explains the over-measurement of the contribution of entry to aggregate TFP changes for the GR and FHK decompositions. The only

**Table 4: Foster, Haltiwanger and Krizan Decomposition**

Aggregate Productivity Change Relative to 1995						
Year	Surviving Firms			Entering Firms	Exiting Firms	All Firms
	Within	Between	Cross			
<i>Labor Productivity (in logs) – Emp. Share Weights</i>						
1996	0.1093	0.0223	-0.0086	-0.0221	0.0311	0.1320
1997	0.2336	0.0324	-0.0139	0.0141	0.0446	0.3108
1998	0.2345	0.0384	-0.0170	0.0333	0.0595	0.3488
1999	0.2874	0.0390	-0.0134	0.0599	0.0575	0.4305
2000	0.3162	0.0418	-0.0133	0.0990	0.0596	0.5033
<i>TFP (in logs) – Value-Added Share Weights</i>						
1996	0.0521	-0.0247	0.0874	0.0057	0.0073	0.1279
1997	0.1480	-0.0229	0.0890	0.0292	0.0185	0.2618
1998	0.1362	-0.0190	0.0900	0.0265	0.0257	0.2593
1999	0.1724	-0.0140	0.1106	0.0662	0.0213	0.3565
2000	0.1911	-0.0225	0.1202	0.0895	0.0229	0.4013

Source: AJPES and own calculations.

difference is that entrants have roughly the same TFP levels (on average) as surviving firms. Thus, the contribution of entrants to productivity change is roughly nil, rather than slightly negative. (This is due to the fact that entrants use capital less intensively than surviving firms; this depresses their labor productivity relative to their TFP in comparison with surviving firms.) Nevertheless, the aggregate TFP of both entrants and surviving firms is growing year over year, leading to the same over-measurement issue that we previously described for labor productivity.

In the case of a growing economy such as Slovenia, the over-measurement of the contribution of entry will be most severe for the FHK decomposition as it uses a lower reference productivity level  $\Phi_1$  than the GR decomposition reference productivity level  $\bar{\Phi}$ . (In both cases, the over-measurement is induced by the fact that the reference productivity levels are below  $\Phi_{S2}$ .) By the same token, this also means that the bias in the measurement of the contribution of exit for the FHK contribution will be less severe than for the GR decomposition. As we have argued in the previous section, the reference productivity level for the contribution of exit should be the aggregate productivity of surviving firms in that same time period,  $\Phi_{S1}$ . The reference productivity level  $\Phi_1$  for the FHK decomposition is closer to this reference point than the reference productivity level  $\bar{\Phi}$ , which is

**Table 5: Dynamic Olley-Pakes Decomposition with Entry and Exit**

Aggregate Productivity Change Relative to 1995					
	Surviving Firms		Entering Firms	Exiting Firms	All Firms
Year	$\Delta\bar{\varphi}_S$	$\Delta cov_S$	$s_{E2}(\Phi_{E2} - \Phi_{S2})$	$s_{X1}(\Phi_{S1} - \Phi_{X1})$	$\Phi_2 - \Phi_1$
<i>Labor Productivity (in logs) – Emp. Share Weights</i>					
1996	0.1094	0.0183	-0.0290	0.0333	0.1320
1997	0.2219	0.0632	-0.0261	0.0518	0.3108
1998	0.2508	0.0483	-0.0228	0.0724	0.3488
1999	0.3174	0.0736	-0.0315	0.0710	0.4305
2000	0.3286	0.1204	-0.0225	0.0768	0.5033
<i>TFP (in logs) – Value-Added Share Weights</i>					
1996	0.1002	0.0178	0.0021	0.0077	0.1279
1997	0.1958	0.0426	0.0028	0.0206	0.2618
1998	0.2145	0.0242	-0.0089	0.0295	0.2593
1999	0.2671	0.0626	0.0017	0.0250	0.3565
2000	0.2758	0.0955	0.0021	0.0279	0.4013

Source: AJPES and own calculations.

substantially greater than  $\Phi_{S1}$ . Thus, the GR decomposition will substantially over-estimate the contribution of exit to productivity growth (as highlighted in tables 3 and 5). As we previously noted, the use of a single reference productivity level for both entry and exit implies that the substantial productivity growth of surviving firms,  $\Phi_{S2} - \Phi_{S1}$ , will be attributed either to the contribution of entry or to that of exit.

For both the GR and FHK decompositions, this implies that the joint contribution of entry and exit will be substantially over-measured – leading to an equally-sized under-measurement of the contribution of surviving firms. Empirically, tables 3-5 reveal that the extent of this under-measurement sums up to 7-10 percentage points of aggregate productivity growth over 5 years. We also see that the lion’s share of this under-measurement pertains to the contribution of market share reallocations among those surviving firms. We find that the contribution of this channel is more than double from that measured by the GR and FHK decompositions. Over the 5 year interval from 1995-2000, we find that those market share reallocations added 12 percentage points to aggregate labor productivity growth and 10 percentage points to aggregate TFP growth over 5 years. Those numbers are now much more substantial, representing a quarter to the productivity



**Table 6: Aggregate Labor Productivity and Employment Shares**

Year		In $t = 1$				
		Surviving Firms		Exiting Firms		All Firms
$t = 1$	$t = 2$	$\Phi_{S1}$	$s_{S1}$	$\Phi_{X1}$	$s_{X1}$	$\Phi_1$
1995	1996	7.4634	0.9334	6.9471	0.0666	7.4289
1995	1997	7.4821	0.8606	7.1013	0.1394	7.4289
1995	1998	7.5037	0.8217	7.0847	0.1783	7.4289
1995	1999	7.5037	0.8106	7.1093	0.1894	7.4289
1995	2000	7.5109	0.7761	7.1450	0.2239	7.4289

  

Year		In $t = 2$				
		Surviving Firms		Entering Firms		All Firms
$t = 1$	$t = 2$	$\Phi_{S2}$	$s_{S2}$	$\Phi_{E2}$	$s_{E2}$	$\Phi_2$
1995	1996	7.6448	0.9570	6.9752	0.0430	7.6160
1995	1997	7.8573	0.8805	7.6392	0.1195	7.8312
1995	1998	7.9395	0.8492	7.7689	0.1508	7.9138
1995	1999	8.0572	0.8022	7.8802	0.1978	8.0222
1995	2000	8.1742	0.7690	8.0603	0.2310	8.1479

Source: AJPES and own calculations.

gains in Slovenian manufacturing.

## 5 Conclusion

In this paper, we proposed an extension of the productivity decomposition method developed by Olley & Pakes (1996). This extension provides an accounting for the contributions of both entry and exit to aggregate productivity changes; and it also breaks down the separate contributions of firm-level productivity shifts and market share reallocations among surviving firms. We argue that the other decompositions that break-down aggregate productivity changes into these same four components introduce some biases in the measurement of the contributions of entry and exit. Furthermore, our proposed decomposition also inherits the attractive properties of the original Olley & Pakes (1996) decomposition that aligns more directly the measured components of aggregate productivity changes within the framework of recent theoretical models featuring heterogeneous firms. We apply our proposed decomposition to the large measured increases of productivity in Slovenian manufacturing during the 1995-2000 period, accounting for the separate contributions of firm-level productivity changes, market share reallocations, entry, and exit. We contrast our results

with those obtained from the other commonly used decompositions that break-down productivity changes into the same four components. Our results highlight that the magnitudes of the measurement bias with those other methods can be substantial over a five year period. In contrast to those other decompositions, we also find that market share reallocations among surviving firms played a much more important role in driving aggregate productivity changes.

In this paper, we have focused on decompositions at the firm-level, which is the finest level of aggregation in our data. However, our decomposition can also be applied at higher levels of aggregation. For example, the unweighted mean-covariance decomposition can be applied to industries as well. This generates a break-down of aggregate productivity into within- and inter-industry components. So long as firms do not switch their industry affiliation, our firm-level decomposition can be nested within the industry level decomposition. There are many other interesting groups of firms to which this decomposition method can be applied.

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## Appendix

### A Alternative Decomposition of Firm Productivity in Levels

In the main text, we developed a decomposition of aggregate productivity as a weighted average of firm productivity measured in logs. Although this is typically the aggregate productivity measure that is reported and decomposed in the literature, it suffers from one potential drawback: it does not directly correspond to a measure that is relevant for aggregate welfare (see Petrin and Levinsohn, 2012, for a further discussion of this topic). Alternatively, one can define aggregate productivity as the weighted average of firm productivity in levels. This aggregate variable has a direct welfare-relevant interpretation, as it captures an average measure of output per unit input. In the case of a single input, the aggregate measure is then simply the aggregate output per unit for the economy (value added per worker in our example with labor productivity).

In this appendix, we show how our decomposition can be applied to this case. The only required modification relates to the covariance measure: we need to define a scale-independent covariance measure that is invariant to proportional changes in measured productivity. Such a measure is obtained simply by dividing firm productivity by aggregate productivity. Our scale independent measure is defined as  $\widetilde{cov} = cov(s, \varphi/\Phi) = cov(s, \varphi)/\Phi$ . As with the covariance used in the main text with productivity measured in logs,  $\widetilde{cov}$  represents the share of aggregate productivity  $\Phi$  that is driven by the correlation between productivity and market share; the remaining share  $\bar{\varphi}/\Phi$  represents the contribution of the productivity distribution – independent of its correlation with market shares.

Using this scale-independent covariance, we can express the relative change in aggregate productivity as:

$$\begin{aligned} \frac{\Phi_2 - \Phi_1}{\bar{\Phi}} &= \frac{\Phi_{S2} - \Phi_{S1}}{\bar{\Phi}} + s_{E2} \frac{\Phi_{E2} - \Phi_{S2}}{\bar{\Phi}} + s_{X1} \frac{\Phi_{S1} - \Phi_{X1}}{\bar{\Phi}} \\ &= \frac{1}{1 - \overline{\widetilde{cov}}_S} \frac{\bar{\Phi}_S}{\bar{\Phi}} \left( \frac{\Delta \bar{\varphi}_S}{\bar{\Phi}_S} + \Delta \overline{\widetilde{cov}}_S \right) + s_{E2} \frac{\Phi_{E2} - \Phi_{S2}}{\bar{\Phi}} + s_{X1} \frac{\Phi_{S1} - \Phi_{X1}}{\bar{\Phi}}, \end{aligned} \quad (\text{A.1})$$

where  $\bar{\Phi} = 1/2(\Phi_1 + \Phi_2)$ ,  $\bar{\Phi}_S = 1/2(\Phi_{S1} + \Phi_{S2})$ ,  $\overline{\widetilde{cov}}_S = 1/2(\overline{\widetilde{cov}}_{S2} + \overline{\widetilde{cov}}_{S1})$  represent time averages over periods 1 and 2. By construction,  $\widetilde{cov} < 1$  and hence  $\overline{\widetilde{cov}}_S < 1$ . This alternate decomposition breaks down aggregate productivity changes (now measured in levels) in a very similar way as the one we introduced in the main text (see (6)). The separation into the contribution of surviving, entering, and exiting firms in the first step proceeds in an identical fashion. We then separate out

the contribution of the surviving firms into the same two components: one reflecting shifts in the distribution of firm productivity in levels via the change in the unweighted mean  $\Delta\bar{\varphi}_S$ , and the other reflecting the change in our scale-independent covariance measure. Unlike our original decomposition, these two terms require a common scaling factor so that they add up to the percentage change in the level of aggregate productivity.<sup>1</sup>

We report the results of this alternate decomposition in Table A.1 using the level of value added per worker and employment weights. This table is thus the counterpart to the top panel of Table 5, highlighting the differences between the use of firm productivity in levels versus logs. We see that the percentage changes in the level of aggregate productivity track the changes in the aggregate of log productivity quite closely, though the former are slightly below the latter by about one percentage point. However, the four separate components of this aggregate productivity change exhibit some larger deviations. Using productivity in levels leads to smaller contributions of exit and of distribution shifts among surviving firms. On the other hand, this alternate decomposition of productivity changes in levels leads to a higher contribution of market share reallocations among surviving firms.

**Table A.1: Dynamic Olley-Pakes Decomposition with Entry and Exit (Productivity in Levels)**

Aggregate Productivity Change Relative to 1995					
Year	Surviving Firms		Entering Firms	Exiting Firms	All Firms
	$\frac{\Delta\bar{\varphi}_S}{(1-\widetilde{cov}_S)\bar{\Phi}}$	$\frac{\Delta\widetilde{cov}_S}{1-\widetilde{cov}_S} \frac{\bar{\Phi}_S}{\bar{\Phi}}$	$s_{E2} \frac{\Phi_{E2}-\Phi_{S2}}{\bar{\Phi}}$	$s_{X1} \frac{\Phi_{S1}-\Phi_{X1}}{\bar{\Phi}}$	$\frac{\Phi_2-\Phi_1}{\bar{\Phi}}$
<i>Labor Productivity (in levels) – Emp. Share Weights</i>					
1996	0.0657	0.0671	-0.0166	0.0200	0.1356
1997	0.1721	0.1173	-0.0230	0.0348	0.3013
1998	0.2383	0.0700	-0.0258	0.0417	0.3307
1999	0.2879	0.1171	-0.0241	0.0402	0.4223
2000	0.2977	0.1642	-0.0195	0.0442	0.4866

Source: AJPES and own calculations.

<sup>1</sup>We can also further decompose the contributions of entry and exit in a similar way:

$$\frac{\Phi_{S1}-\Phi_{X1}}{\bar{\Phi}} = \frac{1}{1-\widetilde{cov}_1} \frac{\Phi_1}{\bar{\Phi}} \left[ \frac{\bar{\varphi}_{S1}-\bar{\varphi}_{X1}}{\Phi_1} + (\widetilde{cov}_{S1}-\widetilde{cov}_{X1}) \right]$$

$$\frac{\Phi_{E2}-\Phi_{S2}}{\bar{\Phi}} = \frac{1}{1-\widetilde{cov}_2} \frac{\Phi_2}{\bar{\Phi}} \left[ \frac{\bar{\varphi}_{E2}-\bar{\varphi}_{S2}}{\Phi_2} + (\widetilde{cov}_{E2}-\widetilde{cov}_{S2}) \right],$$

where  $\widetilde{cov}_1 = s_{X1}\widetilde{cov}_{S1} + (1-s_{X1})\widetilde{cov}_{X1}$  and  $\widetilde{cov}_2 = s_{E2}\widetilde{cov}_{S2} + (1-s_{E2})\widetilde{cov}_{E2}$ .